# DISCRIMINATORS IN COMBINATORY LOGIC AND λ-CALCULUS

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## §0 Introduction

In this paper we consider various systems of Combinatory Logic and $\lambda$ - Calculus augmented by certain "discriminator rules". We will mainly be concerned with consistency and definability problems, but we study also some properties of the resulting reductions.

A discriminator axiom is a reduction axiom of the form

$$(\delta) \qquad \delta XY \longrightarrow \begin{cases} K \text{ if } (X,Y) \in R \\ KI \text{ if } (X,Y) \notin R \end{cases}$$

where $\delta$ is a new constant added to the system considered and R is a binary relation on a specified subset V of the set of all terms of that system. In Combinatory Logic K and KI have the usual meaning, in the $\lambda$-calculus K is $\lambda x \lambda yx$ and KI is $\lambda x \lambda yy$. We will give conditions for V and R which imply the Church-Rosser-theorem for the extended system. We shall make use of the context notation. A context will be a term with one or more holes in it, to be precise

<u>0.1 Definition:</u> The empty context [ ] is a context, if C is a context and M is a term or a context, then CM, MC and $\lambda x C$ are contexts.

This definition works for $\lambda$-Calculus and Combinatory Logic. A context will be denoted by $C \equiv C[\ ,\ ,\ ,\ ]$ where the spaces between the commas indicate the holes in the order in which they appear in C. $C[M_1,\dots,M_n]$ means that $M_1,\dots,M_n$ are substituted for the holes in C.

All other notation is standard. We use $[M/x]N$ for "M substituted for x in N". $\longrightarrow$ always denotes a one-step reduction and we use subscripts to indicate the system in which we work, e.g. $\underset{\beta}{\longrightarrow}$ is $\beta$-reduction defined by $C[R] \underset{\beta}{\longrightarrow} C[R']$ where R is a $\beta$-redex, C any context and R' the contractum of R. Furthermore $\overset{*}{\longrightarrow}$ denotes the reflexive and transitive closure of $\longrightarrow$ . The equality given by $\longrightarrow$ (equivalence relation generated by $\longrightarrow$ ) is denoted by = with the same subscripts as $\longrightarrow$. $\lambda\beta$ denotes the usual $\lambda$-Calculus based on axiom $(\beta)$, CL is Combinatory Logic based on I,K,S as primitive combinators. Additional axioms to $\lambda\beta$ or CL are e.g. indicated by $\lambda\beta+(\delta)$. We sometimes write $\lambda\beta \vdash M = N$, $\lambda\beta \vdash M \longrightarrow N$ for $M \underset{\beta}{=} N$, $M \underset{\beta}{\longrightarrow} N$ and similarly for all other theories. $\equiv$ denotes identity of terms, $M \cong N$ for $\lambda$-terms means that M and N only differ in their bound variables. Throughout this paper we avoid mentioning changes of bound variables. All other notions and notations are standard and can be found in Curry - Feys [4] or in Barendregt [1] .

The first example for a discriminator axiom occurs in Church [3]; there it is proved that the axiom

$$(\delta_{Ch})\ \delta MN \rightarrow \begin{cases} K \text{ if } M \cong N \\ KI \text{ if } M \not\cong N \end{cases} \quad \text{for constant normal forms } M,N$$

can consistently be added to $\lambda\beta$ .

Here constant term means term without free variables (without variables in CL). The same result holds for CL with $\cong$ replaced by $\equiv$. We will derive this result as a corollary to a more general theorem later. The restrictions on M, N in ($\delta_{Ch}$) seem to be quite natural, and indeed some restrictions are necessary as is shown in the first lemma.

<u>0.2 Lemma:</u> CL and $\lambda\beta$ plus the unrestricted discriminator axiom

$$(\delta') \quad \delta MN \rightarrow \begin{cases} K \text{ if } M=N \\ KI \text{ if } M \neq N \end{cases}$$

are inconsistent.

<u>Proof:</u> Let $Y$ in both systems denote the well known paradoxical combinator of Curry and let $F \equiv Y(\delta(KI))$. Then by the fixed point property of $Y$ we have

$$CL\ (\lambda\beta) \vdash F = \delta(KI)F \rightarrow \begin{cases} K \text{ if } F=KI \\ KI \text{ if } F \neq KI \end{cases}$$

In the first case we have $CL(\lambda\beta) \vdash K=KI$ from which we deduce the inconsistency $CL(\lambda\beta) \vdash x=Kxy=KIxy=y$. The second case is obviously contradictory. ⊠

This proof only uses the incompatibility of a fixed point operator and a discriminator. In general F does not have a normal form, so one might try to restrict the discriminator to normal forms, i.e. consider the axiom

$$(\delta'') \quad \delta MN \longrightarrow \begin{cases} K \text{ if } M \equiv N \ (M \cong N) \\ KI \text{ if } M \not\equiv N \ (M \not\cong N) \end{cases} \quad \text{for normal forms } M,N$$

In $\lambda\beta+(\delta'')$ we immediately get the inconsistency

$KI \leftarrow (\lambda x\lambda y(KI))KK \leftarrow (\lambda x\lambda y(\delta xy))KK \rightarrow \delta KK \rightarrow K.$

$CL+(\delta'')$ is consistent as the reader may easily verify using the methods of §1. The reason for this is that in CL the variables behave like constants or free variables. However we loose the substitution property

$$X \rightarrow Y \Rightarrow [M/x]X \rightarrow [M/x]Y$$

because

$[K/x][K/y](\delta xy) \equiv \delta KK \rightarrow K$ and

$[K/x][K/y](KI) \equiv KI$ but not $K \rightarrow KI$.

We finally mention the following undefinability result.

0.3 Theorm: Church's $\delta$ is not definable in the $\lambda$-Calculus, i.e. there is no $\lambda$-term $\delta$ satisfying $(\delta_{Ch})$.

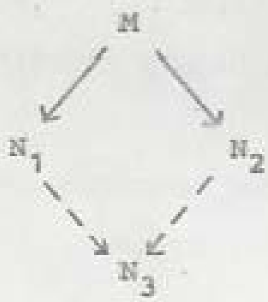
This result was proved independently by Barendregt and Wadsworth, see Barendregt et al. [2] or Hindley- Mitschke [6] for a proof.

## §1 Consistency results

Now we develop conditions for $\delta$-axioms which are sufficient for their consistency with $\lambda\beta$, CL and some other systems. To be more precise: We give conditions for a $\delta$-axiom ($\delta$) from which one can derive the Church-Rosser-theorem for $\lambda\beta+(\delta)$, CL+($\delta$) etc.. For this purpose a lemma discoverd independently by R. Hindley and B. Rosen will be of great help. The Church-Rosser-theorem for a reduction means that $\longrightarrow$ satisfies the property

(CR) If $M \xrightarrow{*} N_1$ and $M \xrightarrow{*} N_2$ then there exists an $N_3$ such that $N_1 \xrightarrow{*} N_3$ and $N_2 \xrightarrow{*} N_3$.

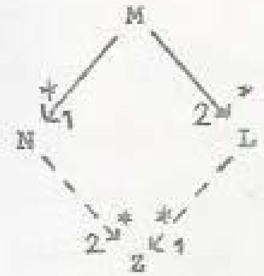
For this we often draw the well known diagram




<u>1.1 Lemma</u>: (Hindley[ 5 ,Thm 8], Rosen[ 9 ,Thm 3.5,3.6] )
Let $\longrightarrow_1$ and $\longrightarrow_2$ be two reduction relations on the same set of terms both satisfying (CR). Then $\longrightarrow = \longrightarrow_1 \cup \longrightarrow_2$ has (CR) if $\longrightarrow_1$ and $\longrightarrow_2$ satisfy

(Com) Whenever $M \longrightarrow_1 N$ and $M \longrightarrow_2 L$ then there exists a $Z$ such that $N \xrightarrow{*}_2 Z$ and $L \longrightarrow_1 Z$ or $L \equiv Z$ ($L \overset{=}{\equiv} Z$).

The abbreviation (Com) indicates that this condition implies that $\longrightarrow_1$ and $\longrightarrow_2$ commute in the sense of the diagram




The proof for this fact is in Rosen[9 , lemma 3.6 ] . For the rest of this paper we restrict ourselves to the $\lambda$-Calculus, but it is easy to verify that most of the results carry over to Combinatory Logic.

Now let $T_\delta$ be the set of all $\lambda$-terms constructed from variables and one additional constant $\delta$, $V \subseteq T_\delta$ be a set of closed terms and $R \subseteq V \times V$ a binary relation on V.

§.2 Definition: (i) The reduction $\xrightarrow[\delta]{}$ assotiated with (V,R) is defined by the axiom

$$(\delta_{(V,R)}) \quad \delta MN \xrightarrow[\delta]{} \begin{cases} K \text{ if } (M,N) \in R \\ KI \text{ if } (M,N) \in V \times V - R \end{cases}$$

(ii) The pair (V,R) is $\delta$-stable iff the following three conditions are satisfied (with $\xrightarrow[\beta\delta]{} = \xrightarrow[\beta]{} \cup \xrightarrow[\delta]{}$ )

(S1) If $M \in V$ and $M \xrightarrow[\beta\delta]{*} M'$ then $M' \in V$.

(S2) If $(M,N) \in R$ , $M \xrightarrow[\beta\delta]{*} M'$ and $N \xrightarrow[\beta\delta]{*} N'$ then $(M',N') \in R$

(S3) If $(M,N) \in V \times V - R$, $M \xrightarrow[\beta\delta]{*} M'$ and $N \xrightarrow[\beta\delta]{*} N'$ then $(M',N') \in V \times V - R$.

The notions of rédex, residual etc. are extended to $\lambda\beta + (\delta_{(V,R)})$ in the usual way, but note that $\delta MN$ is not a redex if $M \notin V$ or $N \notin V$.

The most prominent example for a $\delta$-stable pair is given by: V = the set of all closed normal forms, $R = \cong$.
The $\delta$-reduction associated with this pair is just Church' $\delta$-reduction. In the sequel let $(V,R)$ be a fixed $\delta$-stable pair. Using the Hindley-Rosen-lemma we will show that $\xrightarrow[\beta\delta]{}$ has the Church-Rosser-property.

<u>1.3 Lemma</u>: $\xrightarrow[\delta]{}$ satisfies (CR).
<u>Proof</u>: (CR) for a reduction $\longrightarrow$ says that $\longrightarrow$ commutes with itself. Therefore we show (Com) with $\longrightarrow_1 = \longrightarrow_2 = \xrightarrow[\delta]{}$. Let $M \xrightarrow[\delta]{} N$, $M \xrightarrow[\delta]{} L$ and recall that this means that M contains two redexes $R_1$, $R_2$ and contracting $R_1$ gives N, contracting $R_2$ gives L. We prove (Com) by considering the various possibilities for the relative positions of $R_1$ and $R_2$ in M. Let $R_1'$, $R_2'$ be the contracta of $R_1$, $R_2$.
case 1. $R_1$ and $R_2$ are disjoint. Now $M \equiv C[R_1,R_2]$, consider the diagram

$$\begin{array}{ccc} & M \equiv C[R_1,R_2] & \\ \swarrow & & \searrow \\ N \equiv C[R_1',R_2] & & C[R_1,R_2'] \equiv L \\ \searrow & & \swarrow \\ & Z \equiv C[R_1',R_2'] & \end{array}$$

case 2. $R_1$ is contained in $R_2$. Without loss of generality we assume that $R_2 \equiv \delta C[R_1]X$ and $M \equiv C'[R_2]$. The proof can be taken from the diagram

$$\begin{array}{ccccc} & & M \equiv C'[\delta C[R_1]X] & & \\ & \swarrow & & \searrow & \\ N \equiv C'[\delta C[R_1']X] & & & & C'[R_2'] \equiv L \\ & \searrow & & \swarrow & \\ & & Z \equiv C'[R_2'] & & \end{array}$$

$Z \equiv C'[R_2']$ because the condition of $\delta$-stability makes sure that $\delta C[R_1]X$ and $\delta C[R_1']X$ have the same contractum.

Case 3. $R_2$ is contained in $R_1$. This case is dual to case 2.

No other cases are possible. ⊠

Now we are able to prove

1.4 Theorem: $\xrightarrow[\beta\delta]{}$ has the Church-Rosser-property and therefore $=_{\beta\delta}$ is consistent.

Proof: We check condition (Ĉom) now for $\xrightarrow[\beta]{}$ and $\xrightarrow[\delta]{}$.

Let $M \xrightarrow[\beta]{} N$ by contraction of the $\beta$-redex Q and Q' be the contractum of Q. The $\delta$-redex $\delta AB$ contracted in M to obtain L is (without loss of generality) assumed to have contractum K.

Case 1. $\delta AB$ and Q are disjoint. We have the situation

$$\begin{array}{ccccc} & & M \equiv C[\delta AB, Q] & & \\ & \swarrow_{\beta} & & \searrow^{\delta} & \\ N \equiv C[\delta AB, Q'] & & & & C[K, Q] \equiv L \\ & \searrow_{\delta} & & \swarrow_{\beta} & \\ & & Z \equiv C[K, Q'] & & \end{array}$$

Case 2. Q is in $\delta AB$, say in A. Consider

$$\begin{array}{ccc} & M \equiv C[\delta C'[Q]B] & \\ \swarrow_{\beta} & & \searrow^{\delta} \\ N \equiv C[\delta C'[Q']B] & & C[K] \equiv L \\ \searrow_{\delta} & & \swarrow_{\beta} \\ & C[K] \equiv Z & \end{array}$$

In this case the $\delta$-stability of (V,R) is used to assure that $\delta C'[Q]B$ and $\delta C'[Q']B$ have the same contractum K.

Case 3. $\delta AB$ is in Q, here we have the following two possibilities

a) $M \equiv C'[(\lambda x C[\delta AB])Y]$

b) $M \equiv C'[(\lambda x Y) C[\delta AB]]$

subcase a) is treated in the diagram

$$\begin{array}{ccc} & M \equiv C'[(\lambda x C[\delta AB])Y] & \\ \swarrow_{\beta} & & \searrow^{\delta} \\ N \equiv C'[[Y/x]C[\delta AB] & & C'[(\lambda x C[K])Y] \equiv L \\ \searrow_{\delta} & & \swarrow_{\beta} \\ & Z \equiv C'[[Y/x]C[K]] & \end{array}$$

Her we can conclude $N \xrightarrow[\delta]{} Z$ only because A and B are closed and therefore left unchanged in the reduction $M \xrightarrow[\beta]{} N$.

subcase b) is solved in

$$\begin{array}{ccc} & M \equiv C'[(\lambda x Y) C[\delta AB]] & \\ \swarrow_{\beta} & & \searrow^{\delta} \\ N \equiv C'[[C[\delta AB]/x]Y] & & C'[(\lambda x Y) C[K] \equiv L \\ \searrow_{\delta}^{*} & & \swarrow_{\beta} \\ & Z \equiv C'[[C[K]/x]Y] & \end{array}$$

Here we can conclude only $N \xrightarrow[\delta]{*} Z$ because $C[\delta AB]$ can be substituted for several free occurrences of x in Y in the reduction $M \xrightarrow[\beta]{} N$. So we actually used the full strength of the Hindley-Rosen-lemma.

No other cases are possible. ∎

Some remarks have to be made at this point

Remark 1. Theorem 1.4 respectively its analogue can still be proved if we include the extensionality axiom ($\eta$), one has only to reformulate the definition of $\delta$-stable, in 1.2(ii) one has to replace $\xrightarrow[\beta\delta]{}$ by $\xrightarrow[\beta\eta\delta]{} = \xrightarrow[\beta\eta]{} \cup \xrightarrow[\delta]{}$ .
The extra cases to be considered in the extended proof are trivial.

Remark 2. In the proof of 1.3 and 1.4 not only (Com) was proved but also what Curry calls "strong property (D)" (Curry-Feys [ 4 ,§4B,C]) which inturn is a key to many other nice properties, e.g. Curry's property (E), sometimes also called ($FD^+$). For $\xrightarrow[\beta\delta]{}$ the strong property (D) is the statement:

Let M contain two redexes $R_1$ and $R_2$, $M \xrightarrow[\beta\delta]{} N$ by contracting $R_1$, $M \xrightarrow[\beta\delta]{} L$ by contracting $R_2$. Then there is a Z such that $N \xrightarrow[\beta\delta]{*} Z$ by a development of all residuals of $R_2$ in N, $L \xrightarrow[\beta\delta]{*} Z$ by a development of all residuals of $R_1$ in L. Moreover after both reductions the residuals of all other redexes in M are the same in Z.

As property (D) is true for $\xrightarrow[\beta]{}$ we only have to check the cases where $R_1$, $R_2$ are $\delta$-redexes or one is a $\beta$-redex and the other one a $\delta$-redex, this is just done in the proofs of 1.3 and 1.4 plus the observation that the residuals of the redexes in M are always the same in Z.

Finally we mention some examples for $\delta$-stable pairs (V,R).

Example 1. V = the set of all closed head normal forms, R = the similarity relation between head normal forms. For definitions see Wadsworth[10] or Barendregt [ 1] .

Example 2. V = all closed terms in which $\delta$ does not occur, R = separability of closed terms, i.e.

$(M,N) \in R$ iff $\exists X_1,\ldots,X_k \; (MX_1\ldots X_k \xrightarrow[\beta]{*} K \wedge NX_1\ldots X_k \xrightarrow[\beta]{*} KI)$
The restriction on V is necessary: $\delta KK$ and K are not separable and as K is not separable from itself we would have the unwanted reductions

$\delta(\delta KK)K \xrightarrow[\delta]{} KI$ and

$\delta(\delta KK)K \xrightarrow[\delta]{} \delta(KI)K \xrightarrow[\delta]{} K$ .

In my Habilitationsschrift I asked wether (CR) for $\lambda\beta+(\delta_{(V,R)})$ implies the $\delta$-stability of (V,R). Jan Willem Klop from Utrecht provided the following answer to that problem.

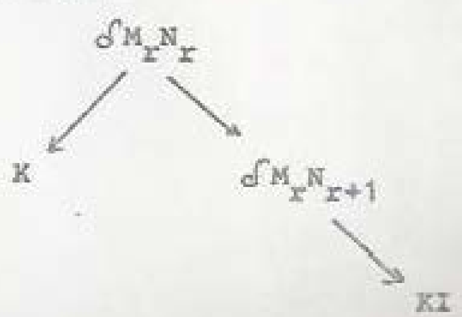
<u>1.5 Theorem</u>: (i) If V is closed under $\xrightarrow[\beta\delta]{*}$ (i.e. (S1) is valid ), then (CR) for $\xrightarrow[\beta\delta]{}$ implies the $\delta$-stability of (V,R).
(ii) (CR) for $\lambda\beta+(\delta_{(V,R)})$ does not imply (S1).

<u>Proof</u>: (i) Suppose that (S2) or (S3) is false, say (S2). Then there exist $M,N \in V$ and $M'$, $N'$ such that $M \xrightarrow[\beta\delta]{*} M'$, $N \xrightarrow[\beta\delta]{*} N'$, $(M,N) \in R$ but $(M',N') \notin R$. Now consider the reductions

$M \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \ldots \longrightarrow M_k \equiv M'$ and

$N \longrightarrow N_1 \longrightarrow N_2 \longrightarrow \ldots \longrightarrow N_l \equiv N'$.

As $(M',N') \notin R$ we must find a number r s.t. $(M_r,N_r) \in R$ and $(M_r,N_{r+1}) \notin R$ , or symmetrically. Now for $M_r$, $N_r$, $N_{r+1}$ we have the reductions



## §2 Discriminators and unsolvables

In $\lambda\beta$ and $\lambda\beta+(\eta)$ the closed head normal forms are exactly the closed solvable terms. Thus for every closed head normal form M we find terms $X_1,\dots,X_k$ such that $MX_1\dots X_k \xrightarrow[\beta]{} N$, where N is a normal form (Wadsworth[10]). When $\delta$-reductions are present we have to define

<u>2.1 Definition</u>: A closed term M is solvable iff there exist terms $X_1,\dots,X_k$ such that $MX_1\dots X_k \xrightarrow[\beta\delta]{} N$ where N is a normal form. An arbitrary term X with free variables $x_1,\dots,x_n$ is called solvable if its closure $\lambda x_1\dots\lambda x_n X$ is solvable.

In $\lambda\beta$ we can replace N in this definition by K, this will fail in general for $\lambda\beta+(\delta_{(V,R)})$, because we can have very strange $\delta$-stable pairs (V,R).
In $\lambda\beta+(\eta)$ one can consistently identify all unsolvable terms (see e.g. Wadsworth[10] or Barendregt et al.[2]). Longo [7] asks wether this remains true in $\lambda\beta$ plus a discriminator on normal forms. In this paragraph a positive answer to this question is given.
We need the following facts about unsolvable terms

1. If U is unsolvable, then also $\lambda x U$ and UM for any M.
2. If U is unsolvable, then also $[M/x]U$ for any x and M, because otherwise $(\lambda x U)M$ would be solvable and therefore $\lambda x U$.

But this contradicts the Church-Rosser-theorem, because $K$ and $KI$ have no common $\underset{\beta\delta}{\twoheadrightarrow}$-reduct.

(ii) Take a $\delta$-stable pair $(V,R)$ and choose an $A \in V$ s.t. there are $A', A''$ with $A' \underset{\beta}{\rightarrow} A \underset{\beta}{\rightarrow} A''$ and $A' \in V$.

Now consider $(V',R') = (V - \{A\}, R - (V-\{A\})^2)$. $(V',R')$ is no longer $\delta$-stable. We note that every reduction in $\lambda\beta + (\delta_{(V,R)})$ can be made into one in $\lambda\beta + (\delta_{(V',R')})$: The only contractions in $\lambda\beta + (\delta_{(V,R)})$ not allowed in $\lambda\beta + (\delta_{(V',R')})$ are of the form

$$\delta AB \rightarrow K \quad ; \quad \delta BA \rightarrow K$$

$$\delta AB \rightarrow KI \quad ; \quad \delta BA \rightarrow KI$$

But e.g. the first one can be replaced by the reduction $\delta AB \underset{\beta}{\rightarrow} A''B \underset{\delta}{\rightarrow} K$ and similarly for the other ones.

As every $\delta_{(V',R')}$-reduction is also a $\delta_{(V,R)}$-reduction, we can easily derive (CR) for $\lambda\beta + (\delta_{(V',R')})$.

$\square$

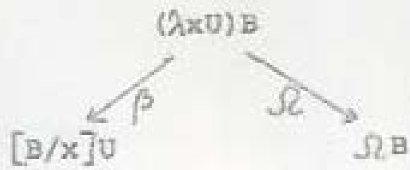
In the rest of this paragraph $\delta$ is the discriminator on closed $\beta\eta\delta$-normal forms. $\Omega \equiv \lambda x(xx)\lambda x(xx)$ is the well known "standard" unsolvable term. $\Omega$-reduction is then defined by the axiom

($\Omega$) $U \xrightarrow[\Omega]{} \Omega$ for any unsolvable term $U$.

We will prove the Church-Rosser-theorem for

$\xrightarrow[\beta\delta\Omega]{} = \xrightarrow[\beta]{} \cup \xrightarrow[\delta]{} \cup \xrightarrow[\Omega]{}$ .

Unfortunately a direct argument via the Hindley-Rosen-lemma does not work because $\xrightarrow[\Omega]{}$ and $\xrightarrow[\beta]{}$ do not commute. A counterexample is

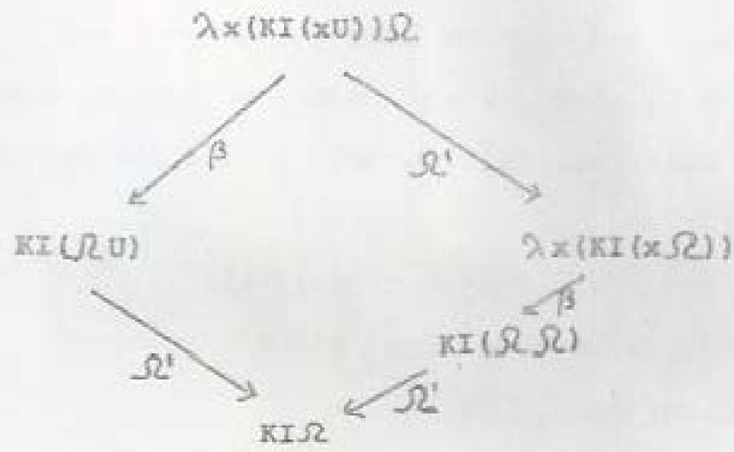



where U is unsolvable. The only possible reductions to fill the diagram are $[B/x]U \xrightarrow[\Omega]{} \Omega$ and $\Omega B \xrightarrow[\Omega]{} \Omega$, but this last contraction should be a $\beta$-step to make commutativity work. To overcome this difficulty in the $\xrightarrow[\beta\eta\Omega]{}$-case Barendregt et al. [2] introduced an auxiliary reduction $\xrightarrow[\Omega']{}$ defined by the axiom

($\Omega'$) $C[U] \longrightarrow C[\Omega]$, if $U$ is a maximal unsolvable subterm of $C[U]$ , i.e. no other subterm of $C[U]$ containing $U$ is unsolvable.

The Church-Rosser-theorem is then proved for $\xrightarrow[\beta\eta]{} \cup \xrightarrow[\Omega']{}$ $\equiv \xrightarrow[\beta\eta\Omega']{}$ via a labelling and underlining technique. I shall give a more direct proof of (CR) for $\xrightarrow[\beta\delta\Omega']{} = \xrightarrow[\beta\delta]{} \cup \xrightarrow[\Omega']{}$ which in my opinion makes better visible the interactions between the different reductions involved. The result will be extended to include $\eta$-reductions.

The Hindley-Rosen-lemma also fails for $\xrightarrow[\beta\delta]{}$ and $\xrightarrow[\Omega']{}$ . Certainly $\xrightarrow[\delta]{}$ and $\xrightarrow[\Omega']{}$ commute because a $\delta$-redex and an unsolvable subterm must be disjoint or the $\delta$-redex is contained in the $\Omega'$-redex. But the diagram




with unsolvable U shows that $\xrightarrow[\beta]{}$ and $\xrightarrow[\Omega']{}$ do not commute.

<u>2.2 Lemma:</u> $\xrightarrow[\Omega']{}$ has the Church-Rosser-property.

<u>Proof:</u> Clear, since two different $\Omega'$-redexes must be disjoint. ☒

<u>2.3 Lemma</u>: Let $M \xrightarrow[\Omega']{*} N$, then any $\beta$- or $\delta$-redex has at most one residual in N.

<u>Proof</u>: This is obvious for $\xrightarrow[\Omega']{}$ and thus for $\xrightarrow[\Omega']{*}$ . ☒

<u>2.4 Lemma</u>: (Postponement of $\Omega'$-contractions)

Let $M \xrightarrow[\beta\delta\Omega']{*} N$, then there exist L and Z such that $M \xrightarrow[\beta\delta]{*} L$, $L \xrightarrow[\Omega']{*} Z$ and $N \xrightarrow[\Omega']{*} Z$.

<u>Proof</u>: By induction on $M \xrightarrow[\beta\delta\Omega']{*} N$. First consider $M \xrightarrow[\Omega']{} M' \xrightarrow[\beta\delta]{} N$.

Let U be the $\Omega'$-redex in M and R the redex in M'.

Case 1. R is a $\delta$-redex.

Take $Z \equiv N$ and use the fact that $\xrightarrow[\delta]{}$ and $\xrightarrow[\Omega']{}$ commute.

Case 2. $R \equiv (\lambda xA)B$ is a $\beta$-redex. We look at the possibilities for the relative positions of R and the contractum of U in M'.

2a) R and $\Omega$ are disjoint; trivial.

2b) $R \equiv \Omega$, trivial because $M' \equiv N$ .

2c) $M' \equiv C'[(\lambda xC[\Omega])B$ .

Now $M \equiv C'[(\lambda xC[U])B$ , we take $L \equiv C'[[B/x]C[U]$ .

Next we look at the residuals of $\Omega$ in N and of U in L

They are contained in maximal unsolvable subterms of N resp. L, contracting these to $\Omega$ reduces N and L to the same term Z.

2d) $M' \equiv C'[(\lambda xA)C[\Omega]$ . Take $L \equiv C'[[C[U]/x]A$ , look at the residuals of U resp. $\Omega$ in L resp. N and proceed like in 2c).

These are all possible cases.

The induction step is easy; use 2.2, 2.3 and the fact that in the case above M reduces to L in at most one step.

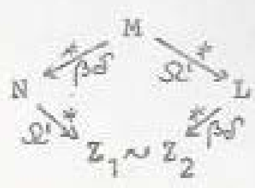
☒

In the last two cases of the preceeding proof $N$ and $L$ only differ in their maximal unsolvable subterms and this was also the case for $KI(\Omega U)$ and $KI(\Omega\Omega)$ in the counterexample to the Hindley-Rosen-lemma. This observation leads to the following definition.

<u>2.5 Definition</u>: $M_1 \sim M_2$ iff there exists a context $C[\ ,\dots,\ ]$ such that $M_1 \equiv C[U_1,\dots,U_k]$, $M_2 \equiv C[V_1,\dots,V_k]$ and $U_1,\dots,U_k$ are exactly the maximal unsolvable subterms of $M_1$, $V_1,\dots,V_k$ are exactly the maximal unsolvable subterms of $M_2$.

Obviously $\sim$ is an equivalence relation. Moreover $M \xrightarrow[\Omega']{*} M'$ implies $M \sim M'$.

<u>2.6 Lemma</u>: $\xrightarrow[\beta\delta]{}$ and $\xrightarrow[\Omega']{}$ commute modulo $\sim$, i.e. whenever $M \xrightarrow[\beta\delta]{*} N$ and $M \xrightarrow[\Omega']{*} L$ then there exist $Z_1$, $Z_2$ such that $N \xrightarrow[\Omega']{*} Z_1$, $L \xrightarrow[\beta\delta]{*} Z_2$ and $Z_1 \sim Z_2$.
In a diagram



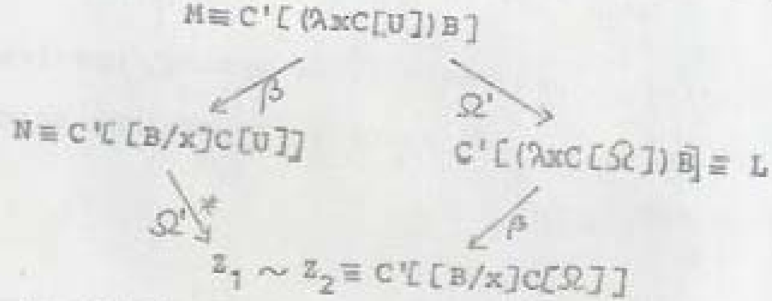

<u>Proof</u>: We proceed by induction on the lengths of the reductions $M \xrightarrow[\beta\delta]{*} N$ and $M \xrightarrow[\Omega']{*} L$.
Case 1. $M \xrightarrow[\beta\delta]{} N$, $M \xrightarrow[\Omega']{} L$, let $R \equiv (\lambda x A)B$ be the $\beta$-redex, $U$ the $\Omega'$-redex in $M$ which are contracted.

1a) R and U are disjoint in M, trivial.

1b) R is contained in U, $U \equiv C[R]$. The residual of U in N is again maximal unsolvable, take $Z_1 \equiv Z_2 \equiv L$.

1c) U is a proper subterm of R, note that the cases $U \equiv A$, $U \equiv \lambda xA$ are impossible because these are not maximal unsolvable.

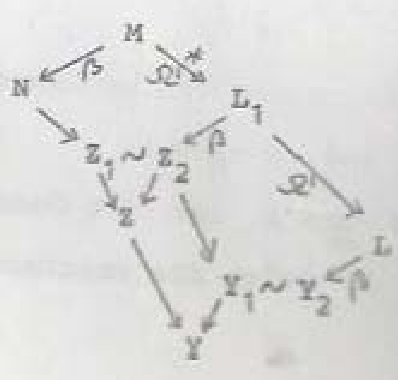
1c1) U is a proper subterm of A. Consider the diagram




Here $Z_1$ is obtained in the following way: Consider the residuals of U in N, these are contained in maximal unsolvable subterms of N, contract these to get $Z_1$. Now $Z_2$ and N only differ in those places where U resp. $\Omega$ stands, therefore $Z_1 \sim Z_2$.

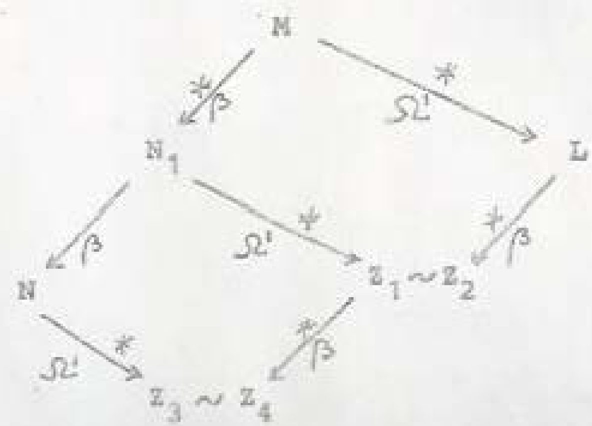
1c2) U is a subterm of B, this case is treated like 1c1.

Case 2. $M \xrightarrow[\beta]{} N$, $M \xrightarrow[\Omega']{*} L$. By induction on the length of the reduction $M \xrightarrow[\Omega']{*} L$. Length one is just case 1. In the induction step consider the diagram



By lemma 2.3 the $\beta$-redex in $M$ has at most one residual in $L_1$ and $L$; $Z_1$, $Z_2$, $Y_1$, $Y_2$ exist by induction hypothesis, from (CR) for $\xrightarrow[\Omega']{}$ we get $Z$ and $Y$, by the remark preceeding lemma 2.6 we have $Y_2 \sim Y$.

Case 3. $M \xrightarrow[\beta]{*} N$, $M \xrightarrow[\Omega']{*} L$, by induction on the length of $M \xrightarrow[\beta]{*} N$. Length one is case 2. In the induction step consider the diagram




in which it is indicated what we know by induction hypothesis. Let $R$ be the $\beta$-redex contracted in the step $N_1 \xrightarrow[\beta]{} N$. We have the following subcases:

3a) $R$ has no residual in $Z_1$, i.e. $R$ is contained in one of the $\Omega'$-redexes in $N_1$ which are contracted in the reduction $N_1 \xrightarrow[\Omega']{*} Z_1$. We are done because $Z_3 \sim Z_4 \equiv Z_1 \sim Z_2$ .

3b) $R$ has a residual $R'$ in $Z_1$.

3b1) $R'$ is contained in a maximal unsolvable subterm of $Z_1$. Now again $Z_3 \sim Z_4 \sim Z_1 \sim Z_2$ .

3b2) $R'$ is not contained in a maximal unsolvable subterm of $Z_1$. Let $Z_1 \equiv C[R']$. By induction on $C$ we find a context $C'$ and a $\beta$-redex $R''$ corresponding to $R'$ with $R'' \sim R'$

and $Z_2 \equiv C'[R'']$ and $R''$ is not contained in a maximal unsolvable subterm of $Z_2$. Now we are done because from the cases 1 and 2 we know that $Z_4$ is the result of contracting $R'$ in $Z_1$, let $Z_5$ be the result of contracting $R''$ in $Z_2$. now $L \xrightarrow[\beta]{*} Z_5$ and obviously $Z_3 \sim Z_4 \sim Z_5$.

Case 4. $M \xrightarrow[\delta]{*} N$, $M \xrightarrow[\Omega']{*} L$ works by a simple induction on the length of the reduction $M \xrightarrow[\delta]{*} N$ because $\delta$- and $\Omega'$-redexes in a term are either disjoint or the $\delta$-redex is a proper subterm of the $\Omega'$-redex.

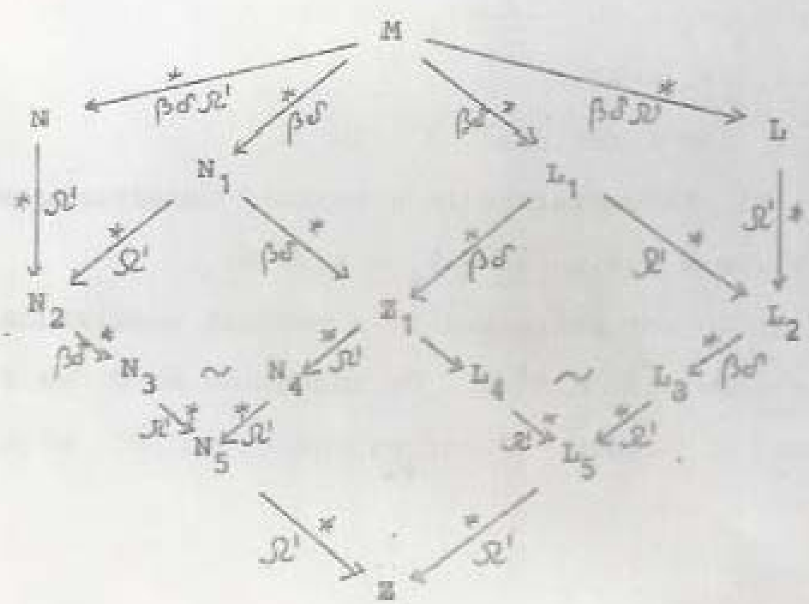
The lemma now follows from the cases 3 and 4 by a simple induction.

⊠

We are now able to prove

<u>2.7 Theorem:</u> $\xrightarrow[\beta\delta\Omega']{}$ has the Church-Rosser-property.

<u>Proof:</u> In the diagram below we find $N_1$, $N_2$, $L_1$, $L_2$ by lemma 2.4; (CR) for $\xrightarrow[\beta\delta]{}$ gives $Z_1$; $N_3$, $N_4$, $L_3$, $L_4$ exist by lemma 2.6; $N_5$, $L_5$ are produced in the obvious way and finally (CR) for $\xrightarrow[\Omega']{}$ gives $Z$.



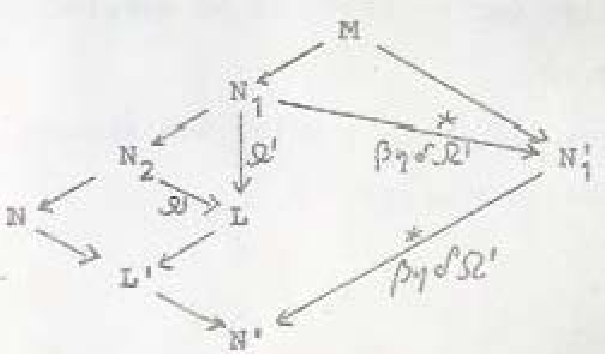

⊠

Once we have 2.7 it is easy to include $\eta$-conversion.

2.8 Theorem: $\xrightarrow[\beta\eta\delta\Omega']{}$ has the Church-Rosser-property.
Proof: By an easy analysis of the possible cases one can see that $\xrightarrow[\eta]{}$ and $\xrightarrow[\beta\delta\Omega']{}$ commute and we get the result from the Church-Rosser-theorem for $\xrightarrow[\eta]{}$. ⊠

Finally like in Barendregt et al.[2] we can drop the restriction that only maximal unsolvable subterms may be reduced to $\Omega$.

2.9 Lemma: Let $M \xrightarrow[\beta\eta\delta\Omega]{*} N$, then there exists an $N'$ such that $M \xrightarrow[\beta\eta\delta\Omega']{*} N'$ and $N \xrightarrow[\beta\eta\delta\Omega']{*} N'$.
Proof: By induction on the number of $\Omega$-steps in the reduction $M \xrightarrow[\beta\eta\delta\Omega]{*} N$. Let $M \xrightarrow[\beta\eta\delta\Omega]{*} N_1 \xrightarrow[\Omega]{} N_2 \xrightarrow[\beta\eta\delta]{} N$ and $N_1 \xrightarrow[\Omega]{} N_2$ be the last $\Omega$-step in the reduction from $M$ to $N$. The $\Omega$-redex $U$ contracted in $N_1$ is contained in a unique $\Omega'$-redex $U'$, $U'$ has a unique residual in $N_2$ which is again maximal unsolvable. Now consider the diagram



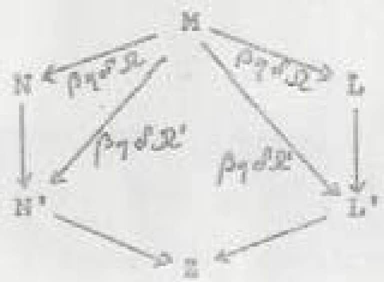
L comes from contracting U in $N_1$ resp. U' in $N_2$, $N_1'$ exists by induction hypothesis and the existence of L' and N' follows from (CR) for $\xrightarrow[\beta\eta\delta\Omega']{}$.

$\boxtimes$

As an immediate consequence we get our main result.

2.10 Theorem: $\xrightarrow[\beta\eta\delta\Omega]{}$ has the Church-Rosser-property.

Proof: Let $N \longleftarrow M \longrightarrow L$, construct according to lemma 2.9 N', L' such that




Now Z as in the diagram exists by (CR) for $\xrightarrow[\beta\eta\delta\Omega']{}$. But any $\xrightarrow[\beta\eta\delta\Omega']{}$ -reduction is also a $\xrightarrow[\beta\eta\delta\Omega]{}$ -reduction.

$\boxtimes$

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Some of the results in this paper are already contained in my Habilitationsschrift.

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